

On canonical quantization of the gauged WZW model with permutation branes

Gor Sarkissian *

Department of Theoretical Physics, Yerevan State University,
Alex Manooghian 1, Yerevan 0025,
Armenia

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Abstract

In this paper we perform canonical quantization of the product of the gauged WZW models on a strip with boundary conditions specified by permutation branes. We show that the phase space of the N -fold product of the gauged WZW model G/H on a strip with boundary conditions given by permutation branes is symplectomorphic to the phase space of the double Chern-Simons theory on a sphere with N holes times the time-line with G and H gauge fields both coupled to two Wilson lines. For the special case of the topological coset G/G we arrive at the conclusion that the phase space of the N -fold product of the topological coset G/G on a strip with boundary conditions given by permutation branes is symplectomorphic to the phase space of Chern-Simons theory on a Riemann surface of the genus $N - 1$ times the time-line with four Wilson lines.

Keywords: conformal field theory, gauged WZW models, permutation branes, topological field theory, D-branes.

* Email address:
gor.sarkissian@ysu.am

1 Introduction

In this paper we continue to investigate the problem of the canonical quantization of the WZW and gauged WZW models with defects and branes, started in the papers [19, 20]. In the paper [19] we addressed the problem of the canonical quantization of the WZW model with defects and permutation branes. In the paper [20] the canonical quantization of the gauged WZW model with defects has been performed. In the present paper we turn to the canonical quantization of the gauged WZW model G/H with permutation branes. It was shown in the paper [19] that the phase space of the N -fold product of WZW models on a strip with boundary conditions given by permutation branes is symplectomorphic to the phase space of Chern-Simons theory on a sphere with N holes times the time-line with two Wilson lines. Using the ansatz for the permutation branes on product of cosets suggested in [18], here we show that the phase space of the N -fold product of the gauged WZW models on a strip with boundary conditions given by permutation branes is symplectomorphic to the phase space of the double Chern-Simons theory on a sphere with N holes times the time-line with G and H gauge fields both coupled to two Wilson lines. For the case of the topological coset G/G we get that the phase space of the N -fold product of the topological coset on a strip with boundary conditions given by permutation branes is symplectomorphic to the phase space of the Chern-Simons theory on a Riemann surface of the genus $N - 1$ times the time-line with four Wilson lines.

2 Bulk WZW model

In this section we review the canonical quantization of the WZW model with compact, simple, connected and simply connected group G on the cylinder $\Sigma = R \times S^1 = (t, x \bmod 2\pi)$ [3, 9, 12]. The world-sheet action of the bulk WZW model is [21]

$$\begin{aligned} S^{\text{WZW}}(g) &= \frac{k}{4\pi} \int_{\Sigma} \text{Tr}(g^{-1} \partial_+ g)(g^{-1} \partial_- g) dx^+ dx^- + \frac{k}{4\pi} \int_B \frac{1}{3} \text{tr}(g^{-1} dg)^3 \quad (1) \\ &\equiv \frac{k}{4\pi} \left[\int_{\Sigma} dx^+ dx^- L^{\text{kin}} + \int_B \omega^{\text{WZ}} \right], \end{aligned}$$

where $x^\pm = x \pm t$. The phase space of solutions \mathcal{P} can be described by the Cauchy data ¹ at $t = 0$:

$$g(x) = g(0, x) \quad \text{and} \quad \xi_0(x) = g^{-1} \partial_t g(0, x). \quad (2)$$

The corresponding symplectic form is [12]:

$$\Omega^{\text{bulk}} = \frac{k}{4\pi} \int_0^{2\pi} \Pi^G(g) dx, \quad (3)$$

where

$$\Pi^G(g) = \text{tr} \left(-\delta \xi_0 g^{-1} \delta g + (\xi_0 + g^{-1} \partial_x g)(g^{-1} \delta g)^2 \right). \quad (4)$$

The δ denotes here the exterior derivative on the phase space \mathcal{P} . It is easy to check that the symplectic form density $\Pi(g)$ has the following exterior derivative

$$\delta \Pi^G(g) = \partial_x \omega^{WZ}(g), \quad (5)$$

what implies closedness of the Ω :

$$\delta \Omega^{\text{bulk}} = 0. \quad (6)$$

The classical equations of motion are

$$\partial_- J_L = 0 \quad \text{and} \quad \partial_+ J_R = 0, \quad (7)$$

where

$$J_L = -ik \partial_+ g g^{-1} \quad \text{and} \quad J_R = ik g^{-1} \partial_- g. \quad (8)$$

The general solution of (7) satisfying the boundary conditions:

$$g(t, x + 2\pi) = g(t, x) \quad (9)$$

is

$$g(t, x) = g_L(x^+) g_R^{-1}(x^-) \quad (10)$$

with $g_{L,R}$ satisfying the monodromy conditions:

$$g_L(x^+ + 2\pi) = g_L(x^+) \gamma, \quad (11)$$

$$g_R(x^- + 2\pi) = g_R(x^-) \gamma \quad (12)$$

¹ Surely we can choose any time slice, but for simplicity we always below take the slice $t = 0$.

with the same matrix γ . Expressing the symplectic form density $\Pi^G(g)$ in the terms of $g_{L,R}$ we obtain:

$$\Pi^G = \text{tr} [g_L^{-1} \delta g_L \partial_x (g_L^{-1} \delta g_L) - g_R^{-1} \delta g_R \partial_x (g_R^{-1} \delta g_R) + \partial_x (g_L^{-1} \delta g_L g_R^{-1} \delta g_R)] . \quad (13)$$

Using (13) and (11), (12) one derives for Ω^{bulk} :

$$\Omega^{\text{bulk}} = \Omega^{\text{chiral}}(g_L, \gamma) - \Omega^{\text{chiral}}(g_R, \gamma) , \quad (14)$$

where

$$\Omega^{\text{chiral}}(g_L, \gamma) = \frac{k}{4\pi} \int_0^{2\pi} \text{tr} (g_L^{-1} \delta g_L \partial_x (g_L^{-1} \delta g_L)) dx + \frac{k}{4\pi} \text{tr} (g_L^{-1} \delta g_L(0) \delta \gamma \gamma^{-1}) . \quad (15)$$

The chiral field g_L can be decomposed into the product of a closed loop in G , a multivalued field in the Cartan subgroup and a constant element in G :

$$g_L(x^+) = h(x^+) e^{i\tau x^+/k} g_0^{-1} , \quad (16)$$

where $h \in LG$, $\tau \in t$ (the Cartan algebra) and $g_0 \in G$. For the monodromy of g_L we find:

$$\gamma = g_0 e^{2i\pi\tau/k} g_0^{-1} . \quad (17)$$

The parametrization (16) induces the following decomposition of $\Omega^{\text{chiral}}(g_L, \gamma)$:

$$\Omega^{\text{chiral}}(g_L, \gamma) = \Omega^{LG}(h, \tau) + \frac{k}{4\pi} \omega_\tau(\gamma) + \text{tr}[(i\delta\tau) g_0^{-1} \delta g_0] , \quad (18)$$

where $\Omega^{LG}(h, \tau)$ is :

$$\Omega^{LG}(h, \tau) = \frac{k}{4\pi} \int_0^{2\pi} \text{tr} [h^{-1} \delta h \partial_x (h^{-1} \delta h) + \frac{2i}{k} \tau (h^{-1} \delta h)^2 - \frac{2i}{k} (\delta\tau) h^{-1} \delta h] dx \quad (19)$$

and $\omega_\tau(\gamma)$ is :

$$\omega_\tau(\gamma) = \text{tr} [g_0^{-1} \delta g_0 e^{2i\pi\tau/k} g_0^{-1} \delta g_0 e^{-2i\pi\tau/k}] . \quad (20)$$

Comparing (14) and (19) to the formulae in appendix C we see that the symplectic phase space of the WZW model on a cylinder coincides with that of Chern-Simons theory on the annulus times the time-line $\mathcal{A} \times R$.

3 Gauged WZW model

Here we review quantization of the gauged WZW model on the cylinder $\Sigma = R \times S^1 = (t, x \bmod 2\pi)$ as it is done in [15].

The action of the gauged WZW model is [2, 10, 11, 16]:

$$S^{G/H}(g, A) = S^{\text{WZW}} + S^{\text{gauge}}, \quad (21)$$

where

$$S^{\text{gauge}} = \frac{k}{2\pi} \int_{\Sigma} L^{\text{gauge}}, \quad (22)$$

$$L^{\text{gauge}}(g, A) = -\text{tr}[-\partial_+ g g^{-1} A_- + g^{-1} \partial_- g A_+ + g A_+ g^{-1} A_- - A_+ A_-]. \quad (23)$$

With the help of the Polyakov-Wiegmann identities:

$$L^{\text{kin}}(gh) = L^{\text{kin}}(g) + L^{\text{kin}}(h) + \text{Tr}(g^{-1} \partial_z g \partial_{\bar{z}} h h^{-1}) + \text{Tr}(g^{-1} \partial_{\bar{z}} g \partial_z h h^{-1}), \quad (24)$$

$$\omega^{\text{WZ}}(gh) = \omega^{\text{WZ}}(g) + \omega^{\text{WZ}}(h) - d\left(\text{Tr}(g^{-1} dg dh h^{-1})\right), \quad (25)$$

it is easy to check that the action (21) is invariant under the gauge transformation:

$$g \rightarrow hgh^{-1}, \quad A \rightarrow hAh^{-1} - dh h^{-1} \quad (26)$$

for $h : \Sigma \rightarrow H$.

The equations of motions are:

$$D_+(g^{-1} D_- g) = 0, \quad \text{Tr}(g^{-1} D_- g T_H) = \text{Tr}(g D_+ g^{-1} T_H) = 0, \quad F(A) = 0, \quad (27)$$

where $D_{\pm} g = \partial_{\pm} g + [A_{\pm}, g]$ and T_H is any element in the H Lie algebra.

The flat gauge field A can be written as $h^{-1} dh$ for $h : R^2 \rightarrow H$ and satisfying:

$$h(t, x + 2\pi) = \rho^{-1} h(t, x) \quad (28)$$

for some $\rho \in H$.

Define $\tilde{g} = hgh^{-1}$. Note that \tilde{g} satisfies

$$\tilde{g}(t, x + 2\pi) = \rho^{-1} \tilde{g}(t, x) \rho. \quad (29)$$

In the terms of \tilde{g} equations (27) take the form:

$$\partial_+(\tilde{g}^{-1} \partial_- \tilde{g}) = 0, \quad \text{Tr}(\tilde{g}^{-1} \partial_- \tilde{g} T_H) = \text{Tr}(\tilde{g} \partial_+ \tilde{g}^{-1} T_H) = 0. \quad (30)$$

The canonical symplectic form density, obtained following the general prescription [4, 5, 12], is given by:

$$\Pi^{G/H}(g, h) = \Pi^G(\tilde{g}) + \partial_x \Psi(h, g), \quad (31)$$

where

$$\Psi(h, g) = \text{tr } h^{-1} dh (g^{-1} dg + dg g^{-1} + g^{-1} h^{-1} dh g). \quad (32)$$

Some properties of the form (32) are summarized in appendix A.

Integrating (31) we get the canonical symplectic form:

$$\Omega^{G/H} = \frac{k}{4\pi} \int_0^{2\pi} \Pi^G(\tilde{g}) dx + \frac{k}{4\pi} \Psi(\rho^{-1}, h g h^{-1}(0)). \quad (33)$$

Collecting (5), (29) and (92) one can show that the form (33) is closed.

Equations (30) can be solved in the terms of the chiral fields:

$$\tilde{g} = g_L(x^+) g_R^{-1}(x^-), \quad \text{Tr}(\partial_y g_L g_L^{-1} T_H) = \text{Tr}(\partial_y g_R g_R^{-1} T_H) = 0 \quad (34)$$

with the monodromy properties:

$$g_L(y + 2\pi) = \rho^{-1} g_L(y) \gamma, \quad g_R(y + 2\pi) = \rho^{-1} g_R(y) \gamma. \quad (35)$$

The monodromy properties (35) imply that the chiral fields $g_{L,R}$ should be written as products of fields as well:

$$g_L = h_B^{-1} g_A, \quad g_R = h_D^{-1} g_C, \quad (36)$$

where $h_B, h_D \in H$ and $g_A, g_C \in G$. The fields in (36) should additionally satisfy:

$$\text{tr}[T_H(\partial_y h_B h_B^{-1} - \partial_y g_A g_A^{-1})] = 0, \quad \text{tr}[T_H(\partial_y h_D h_D^{-1} - \partial_y g_C g_C^{-1})] = 0 \quad (37)$$

and

$$h_B(y + 2\pi) = h_B(y) \rho, \quad g_A(y + 2\pi) = g_A(y) \gamma, \quad (38)$$

$$h_D(y + 2\pi) = h_D(y) \rho, \quad g_C(y + 2\pi) = g_C(y) \gamma. \quad (39)$$

Using (37) one can show:

$$\text{tr}[g_L^{-1} \delta g_L \partial_y (g_L^{-1} \delta g_L)] = \text{tr}[g_A^{-1} \delta g_A \partial_y (g_A^{-1} \delta g_A) - h_B^{-1} \delta h_B \partial_y (h_B^{-1} \delta h_B) + \partial_y (\delta h_B h_B^{-1} \delta g_A g_A^{-1})] \quad (40)$$

and similarly for g_R and h_D, g_C .

Collecting (34), (35), (36), (38), (39), (40) and (13) one can show that

$$\Omega^{G/H} = \Omega^{\text{chiral}}(g_A, \gamma) - \Omega^{\text{chiral}}(g_C, \gamma) - \Omega^{\text{chiral}}(h_B, \rho) + \Omega^{\text{chiral}}(h_D, \rho) \quad (41)$$

Comparing (41) with (14) and remembering that the latter is the symplectic form of the Chern-Simons theory on $\mathcal{A} \times R$, we arrive at the conclusion that the phase space of the gauged WZW model on a cylinder coincides with that of double Chern-Simons theory [15, 17] on $\mathcal{A} \times R$.

4 Permutation branes

Worldvolume Q of the permutation branes on product of cosets $G/H \times G/H$ corresponding to a primary (μ, ν) has been constructed in [18] and have the form:

$$(g_1, g_2) = (cbp, Lp^{-1}L^{-1}), \quad (42)$$

where $p \in G$, $L \in H$, $c \in C_H^\nu$, $b \in C_G^\mu$ and C_G^μ are the conjugacy classes in G :

$$C_G^\mu = \{\beta f_\mu \beta^{-1} = \beta e^{2i\pi\mu/k} \beta^{-1}, \quad \beta \in G\}, \quad (43)$$

where $\mu \equiv \boldsymbol{\mu} \cdot \mathbf{H}$ is a highest weight representation integrable at level k , taking value in the Cartan subalgebra of the G Lie algebra. C_H^ν are the similarly defined conjugacy classes in H . If G and H possess common center, μ and ν should satisfy the selection rule [8].

To write the action one should introduce an auxiliary disc D satisfying the condition $\partial B = \Sigma + D$ and continue the fields g_1 and g_2 on this disc always holding the condition (42).

The action with the boundary condition (42) has the form

$$S_{\mathcal{P}}^{G/H \times G/H} = S^{G/H}(g_1, A_1) + S^{G/H}(g_2, A_2) - \frac{k}{4\pi} \int_D \varpi(L, p, c, b) \quad (44)$$

where

$$\varpi(L, p, c, b) = \Omega^{(2)}(c, b) - \text{tr}((cb)^{-1}d(cb)dpp^{-1}) + \Psi(L, p), \quad (45)$$

where

$$\Omega^{(2)}(c, b) = \omega_\nu(c) - \text{tr}(c^{-1}dcdbb^{-1}) + \omega_\mu(b) \quad (46)$$

and $\omega_\mu(C)$ is defined in (20) and $\Psi(L, p)$ is defined in (32). The form $\varpi(L, p, c, b)$ satisfies the condition:

$$d\varpi(L, p, c, b) = \omega^{WZ}(g_1)|_Q + \omega^{WZ}(g_2)|_Q. \quad (47)$$

One can check that the action (44) is invariant under the gauge transformations:

$$\begin{aligned} g_1 &\rightarrow h_1 g_1 h_1^{-1}, & A_1 &\rightarrow h_1 A_1 h_1^{-1} - dh_1 h_1^{-1}, \\ g_2 &\rightarrow h_2 g_2 h_2^{-1}, & A_2 &\rightarrow h_2 A_2 h_2^{-1} - dh_2 h_2^{-1}, \end{aligned} \quad (48)$$

where $h_1 : \Sigma \rightarrow H$, $h_2 : \Sigma \rightarrow H$. For this purpose note that under (48) the boundary parameters transform in the following way:

$$p \rightarrow h_1 p h_1^{-1}, \quad c \rightarrow h_1 c h_1^{-1}, \quad b \rightarrow h_1 b h_1^{-1}, \quad L \rightarrow h_2 L h_1^{-1}. \quad (49)$$

The gauge invariance follows from the Polyakov-Wiegmann identities and the transformation properties of $\varpi(L, p, c, b)$:

$$\varpi(h_2 L h_1^{-1}, h_1 p h_1^{-1}, h_1 c h_1^{-1}, h_1 b h_1^{-1}) - \varpi(L, p, c, b) = -\Psi(h_1, cbp) - \Psi(h_2, L p^{-1} L^{-1}) \quad (50)$$

which can be obtained using formulae of appendix A.

Consider $G/H \times G/H$ product of coset models on the strip $R \times [0, \pi]$ with boundary conditions on both sides given by the permutation branes:

$$(g_1, g_2)(0) = (C_2 C_1 p_1, L_1 p_1^{-1} L_1^{-1}) \quad (51)$$

$$(g_1, g_2)(\pi) = (C_4 C_3 p_2, L_2 p_2^{-1} L_2^{-1}) \quad (52)$$

Here $C_1 \in C_G^{\mu_1}$, $C_2 \in C_H^{\mu_2}$, $C_3 \in C_G^{\mu_3}$, $C_4 \in C_H^{\mu_4}$, $L_1, L_2 \in H$, $p_1, p_2 \in G$.

The boundary equation of motion resulting from the action (44) at $x = 0$ are:

$$g_1^{-1} D_- g_1 + L_1^{-1} g_2 D_+ g_2^{-1} L_1 = 0 \quad (53)$$

$$C_2^{-1} g_1 D_+ g_1^{-1} C_2 + L_1^{-1} g_2^{-1} D_- g_2 L_1 = 0 \quad (54)$$

$$L_1^{-1} D_t L_1 = 0 \quad C_2^{-1} D_t C_2 = 0 \quad (55)$$

where $D_t = D_+ - D_-$, $D_{\pm} L = \partial_{\pm} L + A_{2\pm} L - L A_{1\pm}$, $D_{\pm} g_1 = \partial_{\pm} g_1 + [A_{1\pm}, g_1]$, $D_{\pm} g_2 = \partial_{\pm} g_2 + [A_{2\pm}, g_2]$, $D_{\pm} C_2 = \partial_{\pm} C_2 + [A_{1\pm}, C_2]$.

Derivation of the equations (53), (54), (55) is outlined in the appendix B. Parameterising again flat gauge fields as

$$A_1 = h_1^{-1} dh_1 \quad A_2 = h_2^{-1} dh_2 \quad (56)$$

one can define as before

$$\begin{aligned}
\tilde{g}_1 &= h_1 g_1 h_1^{-1} & \tilde{g}_2 &= h_2 g_2 h_2^{-1} \\
\tilde{C}_1 &= h_1 C_1 h_1^{-1} & \tilde{C}_2 &= h_1 C_2 h_1^{-1} \\
\tilde{p}_1 &= h_1 p_1 h_1^{-1} & \tilde{L}_1 &= h_2 L_1 h_1^{-1} \\
\tilde{C}_3 &= h_1 C_3 h_1^{-1} & \tilde{C}_4 &= h_1 C_4 h_1^{-1} \\
\tilde{p}_2 &= h_1 p_2 h_1^{-1} & \tilde{L}_2 &= h_2 L_2 h_1^{-1}
\end{aligned} \tag{57}$$

and we have the bulk equations (30) for \tilde{g}_1 and \tilde{g}_2 and boundary equations take the form:

$$\tilde{g}_1^{-1} \partial_- \tilde{g}_1 + \tilde{L}_1^{-1} \tilde{g}_2 \partial_+ \tilde{g}_2^{-1} \tilde{L}_1 = 0 \tag{58}$$

$$\tilde{C}_2^{-1} \tilde{g}_1 \partial_+ \tilde{g}_1^{-1} \tilde{C}_2 + \tilde{L}_1^{-1} \tilde{g}_2^{-1} \partial_- \tilde{g}_2 \tilde{L}_1 = 0 \tag{59}$$

$$\tilde{L}_1^{-1} \partial_t \tilde{L}_1 = 0 \quad \tilde{C}_2^{-1} \partial_t \tilde{C}_2 = 0 \tag{60}$$

Equation (60) implies that \tilde{L}_1 and \tilde{C}_2 are constant along the boundary. Boundary conditions (51) and (52) imply

$$(\tilde{g}_1, \tilde{g}_2)(0) = (\tilde{C}_2 \tilde{C}_1 \tilde{p}_1, \tilde{L}_1 \tilde{p}_1^{-1} \tilde{L}_1^{-1}) \tag{61}$$

$$(\tilde{g}_1, \tilde{g}_2)(\pi) = (\tilde{C}_4 \tilde{C}_3 \tilde{p}_2, \tilde{L}_2 \tilde{p}_2^{-1} \tilde{L}_2^{-1}) \tag{62}$$

Using the chiral decomposition one can solve the boundary equation of motion

$$g_{1R}(y) = \tilde{L}_1^{-1} g_{2L}(-y) m \tag{63}$$

$$g_{2R}(y) = \tilde{L}_1 \tilde{C}_2^{-1} g_{1L}(-y) n \tag{64}$$

Equations (63) and (64) indeed imply (61) with

$$\tilde{p}_1(t) = \tilde{C}_2^{-1} g_{1L}(t) n g_{2L}^{-1}(t) \tilde{L}_1 \tag{65}$$

$$\tilde{C}_1 = \tilde{C}_2^{-1} g_{1L}(t) m^{-1} n^{-1} g_{1L}^{-1}(t) \tilde{C}_2 \tag{66}$$

To have that $\tilde{C}_1 \in C_G^{\mu_1}$ we should require $m^{-1} n^{-1} \equiv R_0 \in C_G^{\mu_1}$.

To satisfy (62) we assume the following monodromy properties of g_{1L} and g_{2L}

$$g_{1L}(y + 2\pi) = \rho_1^{-1} g_{1L}(y) \gamma_1 \quad g_{2L}(y + 2\pi) = \rho_2^{-1} g_{2L}(y) \gamma_2 \tag{67}$$

Now one can show that (62) is satisfied with

$$\tilde{p}_2(t) = \tilde{L}_2^{-1} \tilde{L}_1 \tilde{C}_2^{-1} \rho_1 g_{1L}(\pi + t) \gamma_1^{-1} n g_{2L}^{-1}(\pi + t) \tilde{L}_2 \tag{68}$$

$$\tilde{C}_3 = \tilde{C}_4^{-1} g_{1L}(\pi + t) m^{-1} \gamma_2 n^{-1} \gamma_1 (\tilde{C}_4^{-1} g_{1L}(\pi + t))^{-1} \quad (69)$$

if we require

$$\rho_2^{-1} = \tilde{L}_2 \tilde{L}_1^{-1} \quad (70)$$

and

$$\rho_1^{-1} \tilde{C}_2 \tilde{L}_1^{-1} \tilde{L}_2 = \rho_1^{-1} \tilde{C}_2 \tilde{\rho}_2^{-1} = \tilde{C}_4 \quad (71)$$

where $\tilde{\rho}_2 = \tilde{L}_1^{-1} \rho_2 \tilde{L}_1$.

To have that $\tilde{C}_3 \in C_G^{\mu_3}$ we should require $m^{-1} \gamma_2 n^{-1} \gamma_1 = \tilde{\gamma}_2 R_0 \gamma_1 = R_\pi \in C_G^{\mu_3}$, where $\tilde{\gamma}_2 = m^{-1} \gamma_2 m$.

The monodromies (67) as before can be realized in the terms of the decomposition of the fields g_{1L} and g_{2L} as products:

$$g_{1L} = h_B^{-1} g_A, \quad g_{2L} = h_D^{-1} g_C \quad (72)$$

of the new fields h_B, g_A, h_D, g_C possessing the monodromy properties:

$$h_B(2\pi) = h_B(0) \rho_1, \quad g_A(2\pi) = g_A(0) \gamma_1, \quad (73)$$

$$h_D(2\pi) = h_D(0) \rho_2, \quad g_C(2\pi) = g_C(0) \gamma_2, \quad (74)$$

and satisfying (37).

The symplectic form of product of the gauged WZW models on the strip with boundary conditions specified by the permutation branes can be written using the symplectic form density (31) and the form ϖ :

$$\Omega_{\mathcal{P}}^{G/H} = \frac{k}{4\pi} \left[\int_0^\pi \Pi^{G/H}(g_1, h_1) dx + \int_0^\pi \Pi^{G/H}(g_2, h_2) dx + \varpi(g_1(0), g_2(0)) - \varpi(g_1(\pi), g_2(\pi)) \right]. \quad (75)$$

Substituting in (75) the symplectic form density (31) and using the transformation property (50) we obtain:

$$\Omega_{\mathcal{P}}^{G/H} = \frac{k}{4\pi} \left[\int_0^\pi \Pi(\tilde{g}_1) dx + \int_0^\pi \Pi(\tilde{g}_2) dx + \varpi(\tilde{L}_1, \tilde{p}_1, \tilde{C}_2, \tilde{C}_1) - \varpi(\tilde{L}_2, \tilde{p}_2, \tilde{C}_4, \tilde{C}_3) \right], \quad (76)$$

where \tilde{p}_1 and \tilde{C}_1 defined in (65) and (66) and \tilde{p}_2 and \tilde{C}_3 defined in (68) and (69).

Using (16) one can obtain for (76):

$$\Omega_{\mathcal{P}}^{G/H} = \Omega^{LG}(s_1, \tau_1) + \Omega^{LG}(s_2, \tau_2) - \Omega^{LG}(s_3, \tau_3) - \Omega^{LG}(s_4, \tau_4) + \Omega_1^{\text{bndry}} - \Omega_2^{\text{bndry}} \quad (77)$$

$$\begin{aligned}
\Omega_1^{\text{bdry}} &= \text{tr}[(i\delta\tau_1)f_1^{-1}\delta f_1] + \text{tr}[(i\delta\tau_2)f_2^{-1}\delta f_2] \quad (78) \\
&+ \frac{k}{4\pi} [\omega_{\tau_1}(\gamma_1) + \omega_{\tau_2}(\tilde{\gamma}_2) + \omega_{\mu_1}(R_0) - \omega_{\mu_3}(R_\pi) \\
&- \text{tr}(R_0^{-1}\delta R_0\delta\gamma_1\gamma_1^{-1}) - \text{tr}(\tilde{\gamma}_2^{-1}\delta\tilde{\gamma}_2\delta R_0R_0^{-1}) - \text{tr}(R_0^{-1}\tilde{\gamma}_2^{-1}\delta\tilde{\gamma}_2R_0\delta\gamma_1\gamma_1^{-1})]
\end{aligned}$$

$$\begin{aligned}
\Omega_2^{\text{bdry}} &= \text{tr}[(i\delta\tau_3)f_3^{-1}\delta f_3] + \text{tr}[(i\delta\tau_4)f_4^{-1}\delta f_4] \quad (79) \\
&+ \frac{k}{4\pi} \left[\omega_{\tau_3}(\rho_1) + \omega_{\tau_4}(\tilde{\rho}_2) - \omega_{\mu_2}(\tilde{C}_2) + \omega_{\mu_4}(\tilde{C}_4) \right. \\
&\left. + \text{tr}(\delta\tilde{C}_2\tilde{C}_2^{-1}\delta\rho_1\rho_1^{-1}) + \text{tr}(\tilde{\rho}_2^{-1}\delta\tilde{\rho}_2\tilde{C}_2^{-1}\delta\tilde{C}_2) - \text{tr}(\tilde{C}_2\tilde{\rho}_2^{-1}\delta\tilde{\rho}_2\tilde{C}_2^{-1}\delta\rho_1\rho_1^{-1}) \right]
\end{aligned}$$

Comparing (77) with the corresponding formulae in appendix C we arrive at the conclusion that the phase space of product of coset models on a strip with boundary conditions specified by permutation branes is symplectomorphic to the phase space of the double Chern-Simons theory on an annulus times the time-line and with G and H gauge fields both coupled to two Wilson lines.

5 Permutation branes in topological G/G coset

In this section we discuss permutation branes on the product of topological coset $G/G \times G/G$.

In the previous section we have seen that the phase space of the product of the gauged WZW models on a strip with boundary conditions given by the permutation branes is symplectomorphic to the phase space of the double Chern-Simons theory on an annulus times the time-line with G and H gauge fields both coupled to two Wilson lines. In the case when $G = H$ we arrive at the conclusion that product of topological cosets $G/G \times G/G$ on a strip with boundary conditions given by the permutation branes is equivalent to the Chern-Simons theory on the torus $T^2 = \mathcal{A} \cup (-\mathcal{A})$ times the time-line R with four Wilson lines. This can be verified by a direct calculation. For the case $G = H$ the bulk equations of motion (30) imply that \tilde{g}_1 and \tilde{g}_2 are (t, x) independent.

Therefore one has:

$$\tilde{g}_1(0) = \tilde{C}_2\tilde{C}_1\tilde{p}_1 = \tilde{g}_1(\pi) = \tilde{C}_4\tilde{C}_3\tilde{p}_2 \quad (80)$$

$$\tilde{g}_2(0) = \tilde{L}_1\tilde{p}_1^{-1}\tilde{L}_1^{-1} = \tilde{g}_2(\pi) = \tilde{L}_2\tilde{p}_2^{-1}\tilde{L}_2^{-1} \quad (81)$$

From equations (80) and (81) we get

$$m\tilde{p}_2m^{-1}\tilde{p}_2^{-1}\tilde{C}_3^{-1}\tilde{C}_4^{-1}\tilde{C}_2\tilde{C}_1 = 1 \quad (82)$$

where

$$m = \tilde{L}_1^{-1}\tilde{L}_2 \quad (83)$$

The symplectic form (76) in this case reduces to

$$\Omega_{\mathcal{P}}^{G/G} = \frac{k}{4\pi} \left[\varpi(\tilde{L}_1, \tilde{p}_1, \tilde{C}_2, \tilde{C}_1) - \varpi(\tilde{L}_2, \tilde{p}_2, \tilde{C}_4, \tilde{C}_3) \right], \quad (84)$$

Comparing formulae (82) and (84) with the corresponding formulae in appendix C we arrive at the mentioned symplectomorphism of the product of topological cosets $G/G \times G/G$ on a strip with the boundary conditions given by the permutation branes and that of Chern-Simons theory on the torus times the time-line with four Wilson lines.

6 Final Remarks

The constructions in section 4 and 5 can be easily generalized to N -fold product of coset models G/H . The ansatz for permutation branes has the form:

$$(g_1, \dots, g_N) = (C_2 C_1 p_{N-1} \cdots p_1, L_1 p_1^{-1} L_1^{-1}, \dots, L_{N-1} p_{N-1}^{-1} L_{N-1}^{-1}) \quad (85)$$

where $C_1 \in C_G^{\mu_1}$, $C_2 \in C_H^{\mu_2}$, $p_i \in G$, $L_i \in H$. The ansatz is invariant under the N -fold adjoint action : $g_i \rightarrow h_i g_i h_i^{-1}$, where $h_i : \Sigma \rightarrow H$. Using the Polyakov-Wiegmann identity (25) it is straightforward to check the existence of the two-form ϖ_N satisfying the relation:

$$\sum_1^N \omega^{\text{WZ}}(g_i)|_{\text{brane}} = d\varpi_N \quad (86)$$

Performing the same steps as before we arrive at the conclusion, that the phase space of the N -fold product of the gauged WZW model G/H on a strip with boundary conditions given by permutation branes is symplectomorphic to the phase space of the double Chern-Simons theory on a sphere with N holes times the time-line and with G and H gauge fields both coupled to two Wilson lines. For the special case of the topological coset G/G we get, that the phase space of the N -fold product of the topological cosets G/G on a strip with boundary

conditions given by permutation branes is symplectomorphic to the phase space of Chern-Simons theory on a Riemann surface of genus $N - 1$ times the time-line with four Wilson lines.

A Useful formulae

In this appendix we collect some useful properties of the two-form $\Psi(h, g)$ defined by formula (32).

$$\Psi(hL, p) = \Psi(L, p) + \Psi(h, LpL^{-1}). \quad (87)$$

$$\Psi(Lh^{-1}, hph^{-1}) = \Psi(L, p) - \Psi(h, p). \quad (88)$$

$$\omega_\mu(hCh^{-1}) - \omega_\mu(C) = -\Psi(h, C). \quad (89)$$

$$\Omega^{(2)}(hC_1h^{-1}, hC_2h^{-1}) - \Omega^{(2)}(C_1, C_2) = -\Psi(h, C_1C_2), \quad (90)$$

$$\Psi(h, C_1C_2) = \Psi(h, C_1) + \Psi(h, C_2) + (\tilde{C}_1^{-1}d\tilde{C}_1d\tilde{C}_2\tilde{C}_2^{-1} - C_1^{-1}dC_1dC_2C_2^{-1}), \quad (91)$$

where $\tilde{C}_1 = hC_1h^{-1}$ and $\tilde{C}_2 = hC_2h^{-1}$.

$$\omega^{WZW}(hgh^{-1}) - \omega^{WZW}(g) = -d\Psi(h, g). \quad (92)$$

B Boundary Equations of motion

Computing variation of the action (44) one obtains for the boundary part:

$$\begin{aligned} & \text{tr}[g_1^{-1}\delta g_1(g_1^{-1}\partial_+g_1 + g_1^{-1}\partial_-g_1)]dt + \text{tr}[g_2^{-1}\delta g_2(g_2^{-1}\partial_+g_2 + g_2^{-1}\partial_-g)]dt \\ & + 2\text{tr}[\delta g_1g_1^{-1}A_{1-} - A_{1+}g_1^{-1}\delta g_1 + (\delta g_2g_2^{-1}A_{2-} - A_{2+}g_2^{-1}\delta g_2)]dt + B = 0. \end{aligned} \quad (93)$$

The last term B is one-form satisfying the relation:

$$\text{tr}(g_1^{-1}\delta g_1(g_1^{-1}dg_1)^2) + \text{tr}(g_2^{-1}\delta g_2(g_2^{-1}dg_2)^2) - \delta\varpi = dB. \quad (94)$$

Recalling that the first two terms come from the equation

$$\delta\omega^{WZ} = d[\text{tr}(g^{-1}\delta g(g^{-1}dg)^2)], \quad (95)$$

we see that the existence of the one-form B satisfying (94) is a consequence of the equation (47). Using (42) one can compute B explicitly:

$$\begin{aligned} B = & A_{\mu_1}(C_1) + A_{\mu_2}(C_2) + \text{tr}[C_2^{-1}\delta C_2dC_1C_1^{-1} - \delta C_1C_1^{-1}C_2^{-1}dC_2 - \\ & \delta pp^{-1}(C_2C_1)^{-1}d(C_2C_1) + (C_2C_1)^{-1}\delta(C_2C_1)dpp^{-1} - L^{-1}\delta Lp^{-1}dp + \\ & p^{-1}\delta pL^{-1}dL - L^{-1}\delta Ldpp^{-1} + \delta pp^{-1}L^{-1}dL - L^{-1}\delta Lp^{-1}L^{-1}dLp + \\ & L^{-1}\delta LpL^{-1}dLp^{-1}]. \end{aligned} \quad (96)$$

The one-form $A_\mu(C)$ was defined in [7]:

$$A_\mu(C) = \text{tr}[h^{-1}\delta h(f_\mu^{-1}h^{-1}dhf_\mu - f_\mu h^{-1}dhf_\mu^{-1})], \quad (97)$$

where $C = hf_\mu h^{-1}$, $f_\mu = e^{2\pi i\mu/k}$,
and satisfies:

$$\text{tr}(g^{-1}\delta g(g^{-1}dg)^2)|_{g=C} - \delta\omega_\mu(C) = dA_\mu(C). \quad (98)$$

$A_\mu(C)$ satisfies also another important relation on the time-line:

$$\text{tr}[g^{-1}\delta g(g^{-1}\partial_+g + g^{-1}\partial_-g)]dt + A_\mu(C) = \text{tr}[2\delta hh^{-1}(\partial_+gg^{-1} - g^{-1}\partial_-g)]dt, \quad (99)$$

where $g = C$. Let us explain the meaning of this equation.

The left hand side of the (99) is a particular case of (93) and describes boundary equation of motion of the WZW model with the boundary condition specified by the conjugacy class C , while the right hand side proportional to $J_L + J_R$, what is the condition for the diagonal chiral algebra preservation.

Now, using (42) and (96), one can show by a straightforward calculation, that (93) implies the equations (53), (54), (55) in section 4.

C Symplectic forms of the moduli space of flat connections on a Riemann surface

In this appendix we briefly review the symplectic phase space of the Chern-Simons theory on the three-dimensional manifold of the form $M \times R$, where M is two-dimensional Riemann surface, R is time direction, with n time-like Wilson lines assigned with representations λ_i . It was shown in [6, 22] that the phase space of the Chern-Simons theory in such a situation is given by the moduli space of flat connections on the Riemann surface M punctured at the points z_i where Wilson lines hit M , with the holonomies around punctures belonging to the conjugacy classes $C_G^{\lambda_i} = \eta e^{2\pi i\lambda_i/k} \eta^{-1}$. Therefore denoting holonomies around handles a_j and b_j by A_j and B_j , and around punctures by $M_i \in C_G^{\lambda_i}$ we arrive at the conclusion that the phase space of the Chern-Simons theory is

$$\mathcal{F}_{g,n} = G^{2g} \times \prod_{i=1}^n C_G^{\lambda_i} \quad (100)$$

subject to the relation

$$[B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] M_n \cdots M_1 = I, \quad (101)$$

where

$$[B_j, A_j] = B_j A_j B_j^{-1} A_j^{-1}, \quad (102)$$

and to the adjoint group action.

The symplectic form on $\mathcal{F}_{g,n}$ was derived in [1] and has the form:

$$\Omega_{\mathcal{M}_{g,n}} = \sum_{i=1}^n \Omega_{M_i} + \sum_{j=1}^g \Omega_{H_j}, \quad (103)$$

where

$$\Omega_{M_i} = \frac{k}{4\pi} \omega_{\lambda_i}(M_i) + \frac{k}{4\pi} \text{tr}(K_{i-1}^{-1} \delta K_{i-1} K_i^{-1} \delta K_i), \quad (104)$$

$$\begin{aligned} \Omega_{H_j} &= \frac{k}{4\pi} \Psi(B_j, A_j) + \frac{k}{4\pi} (\text{tr}(K_{n+2j-2}^{-1} \delta K_{n+2j-2} K_{n+2j-1}^{-1} \delta K_{n+2j-1}) \\ &+ \text{tr}(K_{n+2j-1}^{-1} \delta K_{n+2j-1} K_{n+2j}^{-1} \delta K_{n+2j})), \end{aligned} \quad (105)$$

and where

$$K_i = M_i \cdots M_1 \quad i \leq n, \quad (106)$$

$$K_{n+2j-1} = A_j [B_{j-1}, A_{j-1}^{-1}] \cdots [B_1, A_1^{-1}] K_n, \quad (107)$$

$$K_{n+2j} = [B_j, A_j^{-1}] \cdots [B_1, A_1^{-1}] K_n \quad 1 \leq j \leq g.$$

K_0 can be chosen to be equal to the unity element. According to (101) also

$$K_{n+2g} = I. \quad (108)$$

$\omega_\lambda(M)$ and $\Psi(B, A)$ are defined in equations (20) and (32) correspondingly.

It was also proved in [1] that quantization of the moduli space $\mathcal{F}_{g,n}$ with the symplectic form (103) leads to the space of n -point conformal blocks on a Riemann surface of the genus g .

The last piece of information which we need is the symplectic form on the moduli space of flat connections on the punctured sphere with holes $S_{n,m}^2$, where n as before is number of punctures and m is number of holes. It was argued in [6, 14] that the corresponding symplectic form $\Omega_{S_{n,m}^2}$ is given by:

$$\Omega_{S_{n,m}^2} = \Omega_{S_{n+m,0}^2} + \sum_{i=1}^m \Omega_i^{\text{LG}}, \quad (109)$$

where Ω^{LG} is defined in (19) and its geometrical quantization leads to the integrable representation of the affine algebra $\hat{\mathfrak{g}}$ at level k .

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